

Art. 8. General Method to Solve Non-Linear P.D. Equations

CHARPIT'S METHOD

Here we shall be discussing Charpit's general method of solution, which is applicable when the given partial differential equation is not of Type 1 to Type 4 or cannot be reduced to these types

Explanation of Method.

Let given differential equation be

$$f(x, y, z, p, q) = 0 \quad \dots(i)$$

$$\text{We know } dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy \quad (\because z = z(x, y)) \quad \dots(ii)$$

$$\Rightarrow dz = p dx + q dy$$

Now we shall find another relation

$$F(x, y, z, p, q) = 0$$

The auxiliary equations of (x) are

$$\frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dF}{0} \quad \dots (xi)$$

(By using extended form of Lagrange's diff. equation)

Now any solution (integral) of equation (xi) will satisfy Equation (x) consider the simplest relation involving atleast one of p or q for F = 0

Now find p and q from Eqs. (i) and (iii) and put in (ii) i.e., in $dz = p dx + q dy$, which when integrated gives the solution.

Note (i) Equations in (xi)th relation, are known as Charpit's auxiliary equations.

(ii) Remember these in the form as below

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} = \frac{dF}{0}$$

(iii) This method should be applied when differential equation cannot be solved by the methods discussed earlier

OR when it is asked to solve the differential equation by this particular method.

ILLUSTRATIVE EXAMPLES

Example 1. Solve for complete solution by CHARPIT'S METHOD

$$(i) \quad z = p^2 x + q^2 y \quad (ii) \quad q = 3 p^2$$

$$(iii) \quad z = p x + q y + p^2 + q^2 \quad (iv) \quad q = p x + q^2$$

Sol. (i) We are given $z = p^2 x + q^2 y$

$$\Rightarrow \quad p^2 x + q^2 y - z = 0 \quad \dots (i)$$

$$\text{Let } f(x, y, z, p, q) = p^2 x + q^2 y - z = 0$$

$$\Rightarrow \quad \frac{\partial f}{\partial x} = p^2, \quad \frac{\partial f}{\partial y} = q^2, \quad \frac{\partial f}{\partial z} = -1, \quad \frac{\partial f}{\partial p} = 2px, \quad \frac{\partial f}{\partial q} = 2qy \quad \dots (ii)$$

The Charpit's auxiliary equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}}$$

$$\Rightarrow \frac{dx}{-2px} = \frac{dy}{-2qy} = \frac{dz}{-2p^2x - 2q^2y} = \frac{dp}{p^2 - p} = \frac{dq}{q^2 - q} \quad \dots(iii)$$

Taking multipliers as $p^2, 0, 0, 2px, 0$ and then $0, q^2, 0, 0, 2qy$

\therefore each of fraction of (iii) is equal to

$$\frac{p^2 dx + 2px dp}{p^2(-2px) + 2px(p^2 - p)} = \frac{q^2 dy + 2qy dq}{q^2(-2qy) + 2qy(q^2 - q)}$$

$$\Rightarrow \frac{d(p^2x)}{-2p^2x} = \frac{d(q^2y)}{-2q^2y}$$

$$\Rightarrow \frac{d(p^2x)}{p^2x} = \frac{d(q^2y)}{q^2y}$$

Integrating, we get $\log(p^2x) = \log(q^2y) + \log a = \log(q^2ya)$

$$\Rightarrow p^2x = q^2ya \Rightarrow p = \pm \frac{q\sqrt{a}\sqrt{y}}{\sqrt{x}}$$

$$\text{Taking } p = \sqrt{a} \frac{\sqrt{y}}{\sqrt{x}} a$$

Put in (i), we get

$$\begin{aligned} & \frac{ay}{x} q^2x + q^2y = z \\ \Rightarrow & aq^2y + q^2y = z \quad \Rightarrow (a+1)q^2y = z \\ \Rightarrow & q = \pm \frac{\sqrt{z}}{\sqrt{a+1}\sqrt{y}} \end{aligned}$$

$$\text{Take } q = \frac{\sqrt{z}}{\sqrt{a+1}\sqrt{y}}$$

$$\text{Then } p = \sqrt{a} \frac{\sqrt{y}}{\sqrt{x}} \frac{\sqrt{z}}{\sqrt{a+1}\sqrt{y}} = \frac{\sqrt{a}\sqrt{z}}{\sqrt{a+1}\sqrt{x}}$$

We know $dz = p dx + q dy$

$$\Rightarrow dz = \frac{\sqrt{a}}{\sqrt{a+1}} \frac{\sqrt{z}}{\sqrt{x}} dx + \frac{\sqrt{z}}{\sqrt{a+1}\sqrt{y}} dy$$

$$\Rightarrow \frac{dz}{\sqrt{z}} = \frac{\sqrt{a}}{\sqrt{a+1}} x^{-1/2} dx + \frac{1}{\sqrt{a+1}} y^{-1/2} dy$$

Integrating, we get

(iii)

(ii)

$$\begin{aligned} \frac{z^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} &= \frac{\sqrt{a}}{\sqrt{a+1}} \left(\frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right) + \frac{1}{\sqrt{a+1}} \left(\frac{y^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} \right) + c \\ \Rightarrow \sqrt{z} &= \frac{\sqrt{a}}{\sqrt{a+1}} \sqrt{x} + \frac{1}{\sqrt{a+1}} \sqrt{y} + \frac{c}{2} \\ \Rightarrow \sqrt{a+1} \sqrt{z} &= \sqrt{ax} + \sqrt{y} + \frac{c}{2} \sqrt{a+1} \\ \Rightarrow \sqrt{a+1} \sqrt{z} &= \sqrt{ax} + \sqrt{y} + b \text{ say} \end{aligned}$$

which is complete solution of (i)

$$(ii) \text{ We are given } q = 3 p^2 \Rightarrow q - 3 p^2 = 0$$

$$\text{Let } f(x, y, z, p, q) = q - 3 p^2 = 0$$

... (i)

$$\Rightarrow \frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0, \frac{\partial f}{\partial p} = -6p, \frac{\partial f}{\partial q} = 1. \quad \dots (ii)$$

The charpit's auxiliary equations are

$$\begin{aligned} \frac{dx}{-\frac{\partial f}{\partial p}} &= \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} \\ \Rightarrow \frac{dx}{6p} &= \frac{dy}{-1} = \frac{dz}{6p^2 - q} = \frac{dp}{0} = \frac{dq}{0} \quad \dots (iii) \end{aligned}$$

$$\Rightarrow dp = 0 \text{ and } dq = 0 \quad \dots (iv)$$

Integrating $p = a$ and $q = b$

$$\text{Put in (i), we have } b - 3a^2 = 0 \Rightarrow b = 3a^2$$

$$\begin{aligned} \text{We know } dz &= p dx + q dy \\ &= a dx + b dy \end{aligned} \quad (\text{Using (iv)})$$

Integrating

$$z = ax + by + c$$

$$z = ax + 3a^2y + c$$

which is complete integral of (i)

$$(iii) \text{ We are given } z = px + qy + p^2 + q^2 \quad \dots (i)$$

$$\Rightarrow px + qy + p^2 + q^2 - z = 0$$

$$\text{Let } f(x, y, z, p, q) = px + qy + p^2 + q^2 - z = 0 \quad \dots (ii)$$

$$\Rightarrow \frac{\partial f}{\partial x} = p, \frac{\partial f}{\partial y} = q, \frac{\partial f}{\partial z} = -1, \frac{\partial f}{\partial p} = x + 2p, \frac{\partial f}{\partial q} = y + 2q$$

The Charpit's auxiliary equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}}$$

$$\Rightarrow \frac{dx}{-(x+2p)} = \frac{dy}{-(y+2q)} = \frac{dz}{-p(x+2p)-q(y+2q)} = \frac{dp}{p-p} = \frac{dq}{q-q}$$

$$\Rightarrow \frac{dx}{-(x+2p)} = \frac{dy}{-(y+2q)} = \frac{dz}{-(px+qy+2p^2+2q^2)} = \frac{dp}{0} = \frac{dq}{0} \quad \dots(iii)$$

From last fractions ; $dp = 0$ and $dq = 0$

Integrating $p = a$ and $q = b$

We know $dz = p dx + q dy = a dx + b dy$ and

Integrating $z = ax + by + c$

Put in given equation We get $ax + by + c = ax + by + a^2 + b^2$

$$\Rightarrow c = a^2 + b^2$$

($\because p = a, q = b$)

Hence $z = ax + by + a^2 + b^2$ which is complete integral

(iv) We are given $q = px + q^2$

$$\Rightarrow px + q^2 - q = 0 \quad \dots(i)$$

$$\text{Let } f(x, y, z, p, q) = px + q^2 - q = 0 \quad \dots(ii)$$

$$\Rightarrow \frac{\partial f}{\partial x} = p, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0, \frac{\partial f}{\partial p} = x, \frac{\partial f}{\partial q} = 2q - 1. \quad \dots(iii)$$

The charpit's auxiliary equations are

$$\begin{aligned} \frac{dx}{-\frac{\partial f}{\partial p}} &= \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} \\ \Rightarrow \frac{dx}{-x} &= \frac{dy}{-(-2q-1)} = \frac{dz}{-px - q(2q-1)} = \frac{dp}{p+0} = \frac{dq}{0+0} \\ \Rightarrow \frac{dx}{-x} &= \frac{dy}{-(2q-1)} = \frac{dz}{-px - 2q^2 + q} = \frac{dp}{p} = \frac{dq}{0} \quad \dots(iv) \end{aligned}$$

$$\therefore \text{Last fraction of (iv)} \Rightarrow dq = 0$$

$$\Rightarrow q = \text{constant} = a \text{ say} \Rightarrow \boxed{q = a}$$

$$\text{Put in (i)} a = px + a^2 \Rightarrow px = a - a^2$$

or

We know $dz = p dx + q dy$

$$p = \frac{a - a^2}{x}$$

$$\Rightarrow dx = \frac{a - a^2}{x} dx + a dy$$

$$\text{Integrating } z = (a - a^2) \int \frac{dx}{x} + a \int dy + b$$

$$\Rightarrow z = (a - a^2) \log|x| + ay + b$$

which is complete solution of (i)

Example 2. Find the complete solution of

$$pxy + pq + qy = yz$$

Sol. We are given differential equation

$$pxy + pq + qy = yz$$

$$\Rightarrow pxy + pq + qy - yz = 0 \quad \dots(i)$$

$$\text{Let } f(x, y, z, p, q) = pxy + pq + qy - yz$$

$$\Rightarrow \frac{\partial f}{\partial x} = py, \frac{\partial f}{\partial y} = px + q - z, \frac{\partial f}{\partial z} = -y, \frac{\partial f}{\partial p} = xy + q$$

$$\text{and } \frac{\partial f}{\partial q} = p + y \quad \dots(ii)$$

The Charpit's auxiliary equations are

$$\begin{aligned} \frac{dx}{-\frac{\partial f}{\partial p}} &= \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} \\ \Rightarrow \frac{dx}{-(xy+q)} &= \frac{dy}{-(p+y)} = \frac{dz}{-p(xy+q)-q(p+y)} \\ &= \frac{dp}{py+p(-y)} = \frac{dq}{px+q-z-ay} \quad \dots(iii) \end{aligned}$$

From (iii), we have, each member $= \frac{dp}{0}$

(constant)

$$\Rightarrow dp = 0 \Rightarrow p = a$$

Put $p = a$ in (i), we get

$$\begin{aligned} axy + aq + qy - yz &= 0 \\ \Rightarrow q(a+y) &= yz - axy \\ \Rightarrow a &= \frac{yz - axy}{a+y} \end{aligned}$$

We know $dz = p dx + q dy$

$$\begin{aligned} \Rightarrow dz = a dx + \frac{yz - axy}{a+y} dy &\Rightarrow dz - a dx = \frac{y(z - ax)}{a+y} dy \\ \Rightarrow \frac{dz - a dx}{z - ax} &= \frac{y}{a+y} dy \Rightarrow \frac{d(z - ax)}{z - ax} = \frac{(y+a) - a}{a+y} dy = \left(1 - \frac{a}{y+a}\right) dy \end{aligned}$$

Integrating

$$\log |z - ax| = y - a \log |a+y| + b$$

$$\text{or } \log |z - ax| + a \log |a+y| = y + b$$

$$\Rightarrow \log |z - ax| (y + a)^a = y + b$$

$$\begin{aligned}
 &\Rightarrow |(z - ax)(y + a)^a| = e^{y+b} = e^b e^y \\
 &\Rightarrow (z - ax)(y + a)^a = \pm e^b e^y = c e^y \text{ (say) where } c = \pm e^b \\
 &\Rightarrow (z - ax)(y + a)^a = c e^y \\
 &\Rightarrow z - ax = c e^y (y + a)^{-a} \\
 &\quad \text{which is complete integral of (i)}
 \end{aligned}$$

Example 3. Solve $p x + q y = p q$

Sol. We are given the differential equation

$$p x + q y = p q \text{ or } p x + q y - p q = 0 \quad \dots(i)$$

Let $f(x, y, z, p, q) = p x + q y - p q$

$$\Rightarrow \frac{\partial f}{\partial x} = p, \frac{\partial f}{\partial y} = q, \frac{\partial f}{\partial z} = 0, \frac{\partial f}{\partial p} = x - q$$

$$\text{and } \frac{\partial f}{\partial q} = y - p \quad \dots(ii)$$

Now the Charpit's auxiliary equations are

$$\begin{aligned}
 \frac{dx}{-\frac{\partial f}{\partial p}} &= \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}} \\
 \Rightarrow \frac{dx}{-(x-q)} &= \frac{dy}{-(y-p)} = \frac{dz}{-p(x-q) - q(y-p)} \\
 &= \frac{dp}{p} = \frac{dq}{q} \quad \dots(iii)
 \end{aligned}$$

(Using (ii))

From last two members of (iii)

$$\text{we have } \frac{dp}{p} = \frac{dq}{q}$$

Integrating, $\log p = \log q + \log a = \log(qa)$

$$\therefore p = qa$$

Put $p = qa$ in (i), we get $qa x + q y = qa q$

$$\Rightarrow ax + y = aq \Rightarrow q = \frac{ax + y}{a} \quad (\because q \neq 0)$$

Then $p = qa \Rightarrow p = ax + y$

We know $dz = p dx + q dy$

Put values of p and q

$$\Rightarrow dz = (ax + y) dx + \frac{ax + y}{a} dy$$

$$\Rightarrow a dz = a(ax + y) dx + (ax + y) dy = (ax + y)(a dx + dy) = (ax + y) d(ax + y)$$

$$\text{Integrating, } a \int dz = \int (ax + y) d(ax + y) + b$$

$$\Rightarrow az = \frac{(ax + y)^2}{2} + b$$

which is complete solution of (i)

To find Singular Solution

$$\text{Let } F(x, y, z, a, b) = az - \frac{(ax + y)^2}{2} + b \quad \dots(iv)$$

$$\Rightarrow \frac{\partial F}{\partial a} = z - \frac{2}{2}(ax + y)(x) = z - x(ax + y) \quad \text{and} \quad \frac{\partial F}{\partial b} = 1.$$

The singular sol is given by

$$F = 0, \frac{\partial F}{\partial a} = 0, \frac{\partial F}{\partial b} = 0$$

$$\text{But here } \frac{\partial F}{\partial b} = 1 \text{ and } 1 \neq 0$$

\therefore Given equation has no singular solution

To find general solution

Let $b = g(a)$ where g is any arbitrary function

Then (iv) is

$$F(x, y, z, a, g(a)) = az - \frac{1}{2}(ax + y)^2 + g(a)$$

$$\Rightarrow \frac{\partial F}{\partial a} = z - \frac{1}{2}(2)(ax + y)(x) + g'(a)$$

so the general solution is given by

$$ax - \frac{1}{2}(ax + y)^2 + g(a) = 0 \quad \text{and} \quad z - x(ax + y) + g'(a) = 0$$

Example 4. Find the complete solution of

$$p^2 - y^2 q = y^2 - x^2$$

Sol. We are given the differential equation

$$p^2 - y^2 q = y^2 - x^2 \text{ or } p^2 - y^2 q + x^2 - y^2 = 0 \quad \dots(i)$$

$$\text{Let } f(x, y, z, p, q) = p^2 - y^2 q + x^2 - y^2 = 0$$

$$\Rightarrow \frac{\partial f}{\partial x} = 2x, \frac{\partial f}{\partial y} = -2yq - 2y, \frac{\partial f}{\partial z} = 0, \frac{\partial f}{\partial p} = 2p, \quad \dots(ii)$$

$$\frac{\partial f}{\partial q} = -y^2$$

Now the Charpit's auxiliary equations are

$$\frac{dx}{-\frac{\partial f}{\partial p}} = \frac{dy}{-\frac{\partial f}{\partial q}} = \frac{dz}{-p \frac{\partial f}{\partial p} - q \frac{\partial f}{\partial q}} = \frac{dp}{\frac{\partial f}{\partial x} + p \frac{\partial f}{\partial z}} = \frac{dq}{\frac{\partial f}{\partial y} + q \frac{\partial f}{\partial z}}$$

$$\Rightarrow \frac{dx}{-2p} = \frac{dy}{y^2} = \frac{dz}{-p(2p) - q(-y^2)}$$

$$= \frac{dp}{2x} = \frac{dq}{-2y(q+1)}$$

From (iii), we have $\frac{dx}{-2p} = \frac{dp}{2x}$

$$\Rightarrow x \, dx = -p \, dp$$

$$\Rightarrow p \, dp + x \, dx = 0$$

Integrating $\frac{p^2}{2} + \frac{x^2}{2} = \frac{a^2}{2}$ where $\frac{a^2}{2}$ is constant of integration

$$\Rightarrow p^2 = a^2 - x^2 \Rightarrow p = \pm \sqrt{a^2 - x^2}$$

Put in (i)

we get $a^2 - x^2 - y^2 q + x^2 - y^2 = 0$

$$\Rightarrow a^2 - y^2 = y^2 q \Rightarrow q = \frac{a^2 - y^2}{y^2} = \left(\frac{a^2}{y^2} - 1 \right)$$

We know $dz = p \, dx + q \, dy$

Put values of p, q

$$\Rightarrow dz = \pm \sqrt{a^2 - x^2} \, dx + \left(\frac{a^2}{y^2} - 1 \right) \, dy$$

Integrating

$$z = \pm \int \sqrt{a^2 - x^2} \, dx + \int \left(\frac{a^2}{y^2} - 1 \right) dy + b = \pm \left(\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right) + \left(-\frac{a^2}{y} - y \right) + b$$

$$\text{or } z = \pm \left(\frac{x\sqrt{a^2 - x^2}}{2} + \frac{a^2}{2} \sin^{-1} \frac{x}{a} \right) - \frac{a^2}{y} - y + b$$

which is complete solution of (i)

Example 5. Solve $2xz - px^2 - 2qxy + pq = 0$

(D.L.U., Sirsa 2004)

Sol. We are given the differential equation

$$2xz - px^2 - 2qxy + pq = 0 \quad (i)$$

Let $f(x, y, z, p, q) = 2xz - px^2 - 2qxy + pq$

$$\Rightarrow \frac{\partial f}{\partial x} = 2z - 2xp - 2qy, \quad \frac{\partial f}{\partial y} = -2qx, \quad \frac{\partial f}{\partial z} = 2x$$

$$\text{and } \frac{\partial f}{\partial p} = -x^2 + q, \quad \frac{\partial f}{\partial q} = -2xy + p \quad (ii)$$

Now the Charpit's auxiliary equations are